VIBRATIONS

Unit 1 *Week 1*

Sushil Kumar Singh

SGTB Khalsa College
University of Delhi
www.sushilsingh.weebly.com

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Definition

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- The maximum displacement on either side of the equilibrium is the same.
- The energy is conserved and the oscillation continues forever.

EXAMPLES

- a simple pendulum, a mass m swinging at the end of a light rigid rod of length l

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- an electrical circuit, an inductance L connected across a capacitance C carrying a charge q

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where k, the constant of proportionality, is called the *stiffness* and the negative sign shows that the force is acting *against* the direction of increasing displacement and back towards the equilibrium position. This is **Hooke's Law of Elasticity**.

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- If the stiffness k is constant, then the restoring force F when plotted versus x, will produce a straight line and the system is said to be *linear*.
- The equation of motion of such a disturbed system is given by the *dynamic balance* between the forces acting on the system, which by **Newton's Law** is

$$m\ddot{\vec{x}} = -k\vec{x} \quad (2)$$

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$$m\frac{d^2(\vec{X}_1 + \vec{X}_2)}{dt^2} = -k(\vec{X}_1 + \vec{X}_2)$$

General Solution

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- If we rewrite the constants as $A = a\sin\phi$ and $B = a\cos\phi$ where ϕ is a constant angle, then $a = \sqrt{A^2 + B^2}$ and

$$x = a\sin\phi\cos\omega t + a\cos\phi\sin\omega t = a\sin(\omega t + \phi) \quad (3)$$

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- The angle ϕ is called the *Phase constant*. The inclusion of ϕ in the solution allows the motion to be defined from any starting point.
- The velocity of the oscillating system is

$$\dot{x} = \omega a \cos(\omega t + \phi) = \omega a \sin(\omega t + \phi + \pi/2)$$

while the acceleration is

$$\ddot{x} = -\omega^2 a \sin(\omega t + \phi) = \omega^2 a \sin(\omega t + \phi + \pi)$$

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- The total energy at any position x

$$E = U + K = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2 = \frac{1}{2}ka^2 \tag{4}$$

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\langle K \rangle &= \frac{\oint_T \frac{1}{2}m\dot{x}^2 dt}{\oint_T dt} \\
&= \frac{\int_T \frac{1}{2}m\omega^2 a^2 \cos^2(\omega t + \phi) dt}{T} \\
&= \frac{\int_T \frac{1}{2}m\omega^2 a^2 \cos^2\theta d\theta}{\omega T} \\
&= \frac{1}{2}m\omega^2 a^2 \frac{\int_T \cos^2\theta d\theta}{2\pi} \\
&= \frac{1}{2}ka^2 \frac{(4\int_0^{\pi/2}\cos^2\theta d\theta)}{2\pi} \\
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Setting up EOM

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- a frictionless U-tube $(lA\rho)\ddot{x} = -[(2x)A\rho]g$ with $\omega^2 = 2g/l$
- an LC circuit $V_L(= -Ldi/dt = -Ld^2q/dt^2) = V_C(= q/C)$ with $\omega^2 = 1/LC$