VIBRATIONS

Unit 1 *Week 2*

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DEFINITIONS

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- If nearby frequency waves, with same phase and amplitude, superposes Beats are produced. Here the amplitude is modulated. The modulation frequency is equal to the difference in frequencies of the two waves.
- If waves with same frequency and amplitude and a constant phase difference superposes Interference pattern is produced. Here too the amplitude is modulated but the amount of modulation is position dependent. The energy is thus redistributed in space.

Linearity of EOM ⇒ *Superposition*

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- If y_1 and y_2 are two different solutions then

$$\begin{aligned} \alpha \ddot{y}_1 + \beta y_1 &= 0 \\ \alpha \ddot{y}_2 + \beta y_2 &= 0 \\ \Rightarrow \alpha(\ddot{y}_1 + \ddot{y}_2) + \beta(y_1 + y_2) &= 0 \\ \Rightarrow \alpha \frac{d^2(y_1 + y_2)}{dt^2} + \beta(y_1 + y_2) &= 0 \end{aligned}$$

$y_1 + y_2$ is also a solution.

General Superposition

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$$\begin{aligned} y_1 + y_2 &= Im[a_1 e^{i(\omega_1 t + \phi_1)} + a_2 e^{i(\omega_2 t + \phi_2)}] \\ &= Im[e^{i\omega_1 t}(a_1 e^{i\phi_1} + a_2 e^{i(\omega_2 t - \omega_1 t + \phi_2)})] \\ &= Im[e^{i\omega_1 t}(Ae^{i\theta})] \\ &= A \sin(\omega_1 t + \theta) \end{aligned} \quad (2)$$

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where

$$A = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos[(\omega_1 - \omega_2)t + (\phi_1 - \phi_2)]} \tag{3}$$

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- The resultant has modulated amplitude. The phase is varying with time, so the frequency is also not well defined.

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$$\begin{aligned} A &= a\sqrt{2[1+\cos(\phi_1-\phi_2)]} = a\sqrt{2\times 2\cos^2\frac{(\phi_1-\phi_2)}{2}} \\ &= 2a\cos\frac{(\phi_1-\phi_2)}{2} \end{aligned} \tag{7}$$

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- Thus,

$$y = 2a\cos\frac{(\phi_1-\phi_2)}{2}\sin\left(\omega_1 t + \frac{(\phi_1+\phi_2)}{2}\right) \tag{9}$$

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y &= 2a\cos\frac{(\omega_1-\omega_2)}{2}t\sin\left(\omega_1 t+\frac{(\omega_2-\omega_1)}{2}t+\phi\right) \\
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- The frequency $(\omega_1 - \omega_2)/2$ is small if ω_1 and ω_2 are close by. The amplitude seems to be modulated. When we hear such a modulated wave, we intercept the energy at twice the rate at which amplitude changes. So the *Beat frequency* is $\omega_1 - \omega_2$.

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because, for random phases, the average of $\sin^2$ and $\cos^2$ is positive and equals the value $1/2$.

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- The intensity is maximum when the phase difference takes the value $\pm n\pi$ where $n = 0, 1, 2, ...$
- *The Intensity at a given position therefore depends on the Phase Difference betweeen the two superposing waves at that position.*